

August, 1994  
 NORDITA-94/36 P  
 hep-ph/yymmnn

# The Thermal $\beta$ -Function in Yang-Mills Theory

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## Abstract

Previous calculations of the thermal  $\beta$ -function in a hot Yang-Mills gas at the one-loop level have exposed problems with the gauge dependence and with the sign, which is opposite to what one would expect for asymptotic freedom. We show that inclusion of higher-loop effects through a static Braaten-Pisarski resummation is necessary to consistently obtain the leading term, but alters the results only quantitatively. The sign, in particular, remains the same. We also explore, by a crude parameterization, the effects a (non-perturbative) magnetic mass may have on these results.

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# 1 INTRODUCTION

The behaviour of the effective coupling constant  $\alpha_s = g^2/4\pi$  in QCD at high temperature or density has been discussed for a long time, starting with the renormalization group equation (RGE) arguments of Collins and Perry [1] that  $\alpha_s$  decreases logarithmically at high density due to asymptotic freedom. The idea of QCD as a gas of weakly interacting quarks and gluons at high  $T$  originates from this observation. It has later been questioned if it is correct to use the same decreasing  $\alpha_s$  as the renormalized coupling constant when computing general  $n$ -point functions with non-zero external momenta, as the simple scaling assumptions used in [1] do not hold when the external momenta introduce extra dimensionful parameters. The zero temperature RGE can only be expected to be useful when the typical momenta involved scale with the temperature [2, 3]. Also, the argument in [1] assumes that there are no infra-red problems, which are now known to exist [4]. Therefore, several groups have explicitly calculated the  $T$ -dependence of the three-point function in QCD at high  $T$  and used a renormalization group equation, with the temperature and the external momentum  $\kappa$  as scale parameters [5], in order to derive the running of  $\alpha_s$  with  $(T, \kappa)$  [6 – 15]. Even if  $\alpha_s$  was found to decrease logarithmically at high  $T$  it would not be enough to justify an ideal gas approximation of QCD since the typical expansion parameters  $\alpha_s T/\kappa$  and  $\sqrt{\alpha_s} T/\kappa$  still grow at high  $T$ .

Various problems and ambiguities arose when calculating the thermal  $\beta$ -function. It was recognized soon that the dependence of  $\alpha_s(T, \kappa)$  on  $T$  depends strongly on which vertex is chosen to renormalize  $\alpha_s$ , the other vertices being determined by Ward identities [6, 8, 9, 11, 12]. This prescription dependence exists also at  $T=0$  when the momentum-space subtraction is used [16]. There was also some ambiguity in the results which depended on whether the imaginary time formalism (ITF) or a real time formalism was used [11, 12], but this is now better understood [13]. Furthermore, for a given vertex the  $\beta$ -function depends on the momentum prescription and differs, for example, when the collinear and the symmetric points are used, both at zero external energy. Another problem arose in that the result is also gauge fixing dependent [9], which puts into serious question the usefulness of such an approach. It is, in fact, not at all surprising that the  $\beta$ -function shows a gauge dependence when computed using the standard effective action [9] since it is not gauge invariant off-shell. Landsman therefore proposed [10] to use the Vilkovisky–DeWitt effective action [17, 18, 19] to calculate an explicitly gauge independent  $\beta$ -function, though it would still depend on the external momentum prescription. Also a Wilson-loop approach has been used to compute a gauge invariant quark–antiquark potential from which an effective coupling was defined [20]. Such a definition is not directly related to the coupling considered

here.

In this paper we follow the prescription of [10], and use the Vilkovisky–DeWitt effective action to calculate the three–gluon vertex at the static and spatially symmetric point at momentum  $\kappa$  and at temperature  $\tau$  for a  $SU(N)$  Yang–Mills gas. This approach has recently been used in [14, 15] where the one–loop  $\beta$ –function was calculated and the scaling in  $\tau$  and  $\kappa$  was analysed. The choice of the static renormalization point can be partially motivated by the fact that in the ITF it is only the zero Matsubara frequency modes that are soft and need resummation (see Section 3). It also eliminates the problem of choosing between analytic continuations (retarded/advanced or time/anti-time ordered) which have different soft contributions [21].

Since the  $\beta$ –function here is linearly related to the two–point function by a Ward identity one might naively expect that it would have a high temperature dependence like  $\tau^2/\kappa^2$ . However, at the static point there is a cancellation and it is found that at one loop [10, 14]

$$\tau \frac{dg}{d\tau} = \frac{g^3}{8\pi^2} N \frac{21\pi^2}{16} \frac{\tau}{\kappa} . \quad (1)$$

The leading linear contribution does not come from the hard part of the loop integral, responsible for a  $\tau^2/\kappa^2$ –term, but from soft loop momenta. Therefore, in the spirit of the Braaten–Pisarski resummation scheme [23], it is not consistent to stop the calculation at the one–loop order for soft internal momenta, but the resummed propagator and vertices must be used to get the complete leading contribution. The main purpose of this paper is to perform the resummed one–loop calculation and analyse the new result. We do not include any fermion contribution since it is subleading at high  $T$ .

## 2 PERTURBATIVE EXPANSION OF THE $\beta$ –FUNCTION

The RGE with the temperature and momentum  $(\tau, \kappa)$  as parameters was first derived in [5] using the fact that the renormalized  $n$ –point functions are formally independent of the renormalization condition. We would like to relate this RGE to a direct calculation of the derivative of the three–gluon function. Let us first fix the notation and work in the Landau gauge in this section — it can be shown that results in this gauge, using the background field method, coincide with those results of the Vilkovisky–DeWitt

effective action [19]. The inverse of the full propagator is

$$\begin{aligned}
(-i\Delta^{-1})_{\mu\nu}^{ab} &= \delta^{ab}(g_{\mu\nu}P^2 - P_\mu P_\nu) - \delta^{ab}\Pi_{\mu\nu}(P) , \\
\Pi_{\mu\nu}(P) &= A_{\mu\nu}\Pi^T(P) + B_{\mu\nu}\Pi^L(P) , \\
A_{\mu\nu} &= g_{\mu\nu} - B_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} , \\
B_{\mu\nu} &= \frac{V_\mu V_\nu}{V^2} , \quad V_\mu = P^2 U_\mu - U \cdot P P_\mu .
\end{aligned} \tag{2}$$

The four-velocity of the heat-bath is given by  $U_\mu = (1, 0, 0, 0)$ . The Ward identities in the Vilkovisky–DeWitt effective action are particularly simple due to the off-shell gauge invariance, and the spatial part of the three-gluon vertex, for static and symmetric external momenta, can be related to the transverse part of the polarization tensor through

$$\Gamma_{ijk}^{abc}(\tau, \kappa) = g f^{abc} \left\{ [g_{ij}(p-q)_k + g_{jk}(q-r)_i + g_{ki}(r-p)_j] \left( 1 + \frac{\Pi^T(\tau, \kappa)}{\kappa^2} \right) + \dots \right\} , \tag{3}$$

where  $P_\mu = (p_0 = 0, \mathbf{p})$ ,  $p^2 = |\mathbf{p}|^2$  etc.,  $p^2 = q^2 = r^2 = \kappa^2$ , and the dots stand for terms orthogonal to  $p_i$ ,  $q_j$  and  $r_k$ . When  $p_0 = 0$  we have  $2\Pi^T = -\sum_i \Pi_{ii}$  using the Minkowski metric. The wavefunction and coupling constant renormalizations ( $A_\mu^a \rightarrow Z_3^{1/2} A_{R\mu}^a$ ,  $g \rightarrow Z_3^{-1/2} g_R$ ) are performed at  $(\tau, \kappa)$  so that

$$\begin{aligned}
(-i\Delta^{-1}(\tau, \kappa))_{ij} &= (\delta_{ij}p^2 - p_i p_j) \Big|_{p^2=\kappa^2} , \\
\Gamma_{ijk}^{abc}(\tau, \kappa) &= g_R(\tau, \kappa) f^{abc} \left\{ [g_{ij}(p-q)_k + \text{cycl.}] + \dots \right\} \Big|_{p^2=q^2=r^2=\kappa^2} .
\end{aligned} \tag{4}$$

We now define an effective coupling constant  $g(T, \kappa)$  at another temperature  $T$  by

$$\Gamma_{ijk}^{abc}(T, \kappa) Z_3^{3/2}(T, \tau) \equiv g(T, \kappa) f^{abc} \left\{ [g_{ij}(p-q)_k + \text{cycl.}] + \dots \right\} , \tag{5}$$

where  $Z_3^{1/2}(T, \tau) = Z_3^{1/2}(T)/Z_3^{1/2}(\tau)$  is the rescaling of the field which is required in order to keep the normalization of the two-point function. In this way  $g(\tau, \kappa)$  measures the non-linearity of the theory. It is now straightforward to derive

$$\beta_\tau \equiv \tau \frac{dg(\tau, \kappa)}{d\tau} = -\frac{g}{2\kappa^2} T \frac{d\Pi^T(T, \kappa)}{dT} \Big|_{T=\tau} , \tag{6}$$

using the renormalization condition in Eq.(4). Similarly we find

$$\beta_\kappa \equiv \kappa \frac{dg(\tau, \kappa)}{d\kappa} = -\frac{g}{2} |\mathbf{p}| \frac{d}{d|\mathbf{p}|} \left( \frac{\Pi^T(\tau, |\mathbf{p}|)}{|\mathbf{p}|^2} \right) \Big|_{|\mathbf{p}|=\kappa} . \tag{7}$$

For a perturbative calculation of  $\beta_\tau$  ( $\beta_\kappa$  can be treated similarly) we need only the expression for  $\Pi^T(T, \kappa)$  in the vicinity of the renormalization point, and we then use this in fixing the initial condition for the RGE. If the renormalization point is chosen appropriately, we could expect reliable results in some regime around  $(\tau, \kappa)$  from a one-loop computation of  $\Pi^T(T, \kappa)$ .

We know that the leading hard thermal loops are non-local and if we want to include them through some resummation we also need non-local counter terms. Therefore, we write the action as

$$\mathcal{L} = -\frac{1}{4}\text{tr}(F^2) + \frac{1}{2}A_\mu(-P)\pi^{\mu\nu}A_\nu(P) - \frac{1}{2}A_\mu(-P)\pi^{\mu\nu}A_\nu(P), \quad (8)$$

and associate the first  $\pi^{\mu\nu}$  with the “bare” propagator and consider the other one as a counter term. Then, we impose on the transverse and longitudinal part of  $\pi^{\mu\nu}$  the one-loop hard thermal loop form

$$\begin{aligned} \pi^L(p_0, p) &= g^2(\tau, \kappa) \frac{\tau^2 N}{3} \left(1 - \frac{p_0^2}{p^2}\right) \left[1 - \frac{p_0}{2p} \ln \left|\frac{p_0 + p}{p_0 - p}\right|\right], \\ \pi^T(p_0, p) &= g^2(\tau, \kappa) \frac{\tau^2 N}{6} \left[\frac{p_0^2}{p^2} + \left(1 - \frac{p_0^2}{p^2}\right) \frac{p_0}{2p} \ln \left|\frac{p_0 + p}{p_0 - p}\right|\right]. \end{aligned} \quad (9)$$

It is enough to introduce the momentum dependence in  $\pi^{L,T}$  from hard thermal loops in order to resum the leading  $g^2\tau^2$  contribution. The effective propagator, defined by

$$(-iD^{*-1})_{\mu\nu}^{ab} = \delta^{ab}(g_{\mu\nu}P^2 - P_\mu P_\nu) - \delta^{ab}\left(A_{\mu\nu}\pi^T(p_0, p) + B_{\mu\nu}\pi^L(p_0, p)\right), \quad (10)$$

has an explicit  $\tau$  dependence which leads to a  $\tau$ -dependent UV divergence already at the one-loop level, since the  $iD_{\mu\nu}^{*ab}$  contains contributions from an infinite sum of higher order diagrams. This  $\tau$ -dependence disappears when all diagrams to a given order are included [24] and thus the problem can be pushed to arbitrarily high order by performing the renormalization to higher order. In our approach we only have to assume that this had been carried out at some  $\tau$  when renormalizing  $\Pi^T(\tau, \kappa)$ . After taking the  $T$  derivative and the limit  $T \rightarrow \tau$  everything is finite. We also note that vertices have hard thermal loop corrections so they should be treated in a similar manner by adding and subtracting the effective vertices in Eq.(8), but we do not need them in the approximation we are using (see Section 3).

Let us now analyse the perturbative calculation of  $\beta_\tau$  using the renormalized Lagrangian in Eq.(8). The high temperature expansion of  $\Pi^T(T, \kappa)$  is an expansion in  $g^2T^2/\kappa^2$  and  $g^2T/\kappa$ . The  $g^2T^2/\kappa^2$  only comes from the hard thermal loops and for each such diagram there is a corresponding counter term  $g^2\tau^2/\kappa^2$  with the opposite

sign generated by the last term in Eq.(8). This is so because the counter term is chosen to be exactly the hard thermal loop contribution. The  $\beta_\tau$ -function is finally computed as the derivative of  $\Pi^T(T, \kappa)$  with respect to  $T$  at  $T = \tau$ . If a diagram contains two or more hard thermal loops the leading  $g^2 T^2/\kappa^2$  and  $g^2 \tau^2/\kappa^2$  terms factor out in such a way that after taking the derivative and  $T \rightarrow \tau$  they cancel. It then follows that in the perturbative expansion of  $dg(\tau, \kappa)/d\tau$  at most one hard thermal loop contribute in each diagram and it is in fact an expansion in  $g^2 \tau/\kappa$  only. The cancellations are identical to what was found for the  $\phi^4$ -model in [25] except that here we must use momentum dependent counter terms since the hard thermal loops are non-local. Also the usual way of simply using improved propagators to do loop calculations, without the RGE, does indeed resum the leading powers of  $g^2 \tau^2/\kappa^2$ . The difference is here that  $g$  itself is not a fixed zero temperature parameter but it is defined through the solution to the RGE. Therefore, the expansion is really in powers of  $g^2(\tau, \kappa)\tau/\kappa$  and its value depends on the solution of the temperature renormalization group equation. The possibility of performing a perturbative expansion at high  $\tau$  for fixed  $\kappa$  depends on whether this combination increases or decreases at large  $\tau$ .

Before doing any actual computation of the resummed  $\beta$ -function it is interesting to discuss what kind of new terms one can expect and what their consequences would be. Let us therefore write

$$\tau \frac{dg}{d\tau} = \beta_\tau^{(0)} + \beta_\tau^L + \beta_\tau^T = g^3 \frac{\tau}{\kappa} (c_0 + c_1 \frac{g\tau}{\kappa} + c_2 \frac{g^2 \tau}{\kappa}) , \quad (11)$$

assuming a high  $\tau$  expansion ( $\tau \gg \kappa$ ). The contribution from hard thermal loops is denoted by  $\beta_\tau^L$  since it is generated by a longitudinal mass (see Section 3). In the expansion of  $\beta_\tau$  in Eq.(1) there is no contribution from any hard thermal loop and we expect that the use of resummed propagators will supply the  $c_1$  term of relative order  $g\tau/\kappa$ . The inclusion of  $\beta_\tau^T$  from a transverse “magnetic mass” of order  $g^2 \tau$ , as discussed below, would generate the  $c_2$  term. We assume that the initial condition is given at a temperature  $\tau_0 \gg \kappa$  while we still have  $g^2(\tau_0, \kappa)\tau_0 \ll \kappa$  so that we can do a consistent perturbation expansion in  $g^2 \tau_0/\kappa$ . As  $\tau$  increases the solution to the RGE determines whether  $g^2(\tau, \kappa)\tau/\kappa$  stays small enough for the perturbative expansion to remain valid. With the positive sign in Eq.(1) for the bare one-loop  $\beta$ -function the coupling constant diverges at some  $\tau$  implying that the expansion breaks down. If the sign had been negative the solution would go like  $g(\tau, \kappa) \sim (\tau/\kappa)^{-1/2}$ , implying that  $g\tau/\kappa$  increases and has to be resummed while  $g^2 \tau/\kappa$  goes to a constant and can be treated perturbatively if it is not too large. In the present case the bare one-loop calculation gives a divergent  $\alpha_s$  but resummation of  $g\tau/\kappa$  terms may change this. In particular, in Eq.(11), if  $c_1$  is negative at large  $g\tau/\kappa$  it dominates over the constant

term and the asymptotic form of  $g(\tau, \kappa)$  is  $(\tau/\kappa)^{-2/3}$ . The factor  $g\tau/\kappa$  still increases and needs to be resummed (as done with the momentum dependent counter terms in the temperature renormalization group equation) but the  $g^2\tau/\kappa$  terms actually go to zero and the exact high  $\tau$  limit would be under control. We have found (see Section 3) that  $c_1$  is actually zero but there is a correction to the constant  $c_0$ , though it does not change the sign of  $\beta_\tau$  for large  $g\tau/\kappa$ .

It is also interesting to see what happens if a magnetic mass is present; although such an effect is believed to be non-perturbative, we could crudely mimic such a term perturbatively by introducing some constant  $m_T \sim g^2\tau$  by hand as the position of the pole of the static transverse mode. Assuming that  $c_2$  is negative and dominates we find that  $g(\tau, \kappa) \sim (\tau/\kappa)^{-1/2}$ . It may thus be inconsistent to assume that  $g^2\tau/\kappa$  is large since it goes to a constant, and one would have to solve the renormalization group equation with the full  $(\tau, \kappa)$  dependence. We found that  $c_2$  is gauge dependent and positive in the Landau gauge. Again, even if this correction would make  $\beta_\tau$  negative it is not consistent to separate out  $g^2\tau/\kappa$  and subleading constants since they all go to constants.

To summarize, a negative  $c_1$  term would cure the problem of a divergent perturbative expansion of  $\beta_\tau$ , but the actual result shows only corrections to the  $c_0$  and the sign remains positive leading to a divergent  $g(\tau, \kappa)$  at some finite  $\tau$ .

### 3 THE RESUMMED ONE-LOOP CALCULATION

To find the  $\beta$ -function in the scheme described in Section 1 we need to compute the transverse part of the polarization tensor at one-loop using the effective propagators and vertices including hard thermal loop corrections. We shall perform the calculation in an arbitrary covariant background field gauge for comparison with other results and to see which terms are gauge independent, though the Vilkovisky–DeWitt approach prescribes the Landau gauge. The Feynman rules in the background gauge can be found in [26] and a one-loop calculation of the polarization tensor at finite  $T$  was performed in [27]. Let us start with  $\beta_\tau$ -function without resummation in a general covariant background gauge, parametrized with  $\xi$ . The Landau gauge  $\xi = 0$  was considered in [14, 15] and the Feynman gauge  $\xi = 1$  in [28]. We can extract the result for general  $\xi$  from the calculation in [27]. Furthermore, the leading  $\tau/\kappa$  comes from the IR dominant part of the loop and is determined by the  $n = 0$  Matsubara frequency. For diagrams that are UV-convergent we can extract the linear  $\tau/\kappa$  term by simply restricting the sum to  $n = 0$ . Diagrams that are not UV-convergent have to be summed over all  $n$ . For the integrals we are dealing with it turns out that if the diagram is only logarithmically

divergent it is in fact enough to take the  $n = 0$  term to get the correct leading real part. There is an example in [27] where this does not work for the imaginary part. The expression needed for the one-loop polarization tensor in a general background field gauge, including the ghost contributions, is [27] (we are using a different sign convention than [27])

$$\begin{aligned} \Pi_{\mu\nu}(K) = & -g^2 N T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left[ g_{\mu\nu} i D_\alpha^\alpha(P) - \left(1 - \frac{1}{\xi}\right) i D_{\mu\nu}(P) \right. \\ & + \frac{2P_\mu P_\nu - P_\mu Q_\nu - Q_\mu P_\nu}{P^2 Q^2} - \frac{2g_{\mu\nu}}{P^2} \\ & \left. + \frac{1}{2} \Gamma_{\alpha\beta\mu}(P, Q, K) D^{\alpha\alpha'}(P) D^{\beta\beta'}(Q) \Gamma_{\alpha'\beta'\nu}(P, Q, K) \right], \end{aligned} \quad (12)$$

where  $P + Q + K = 0$  and the bare three-point vertex is

$$\Gamma_{\alpha\beta\mu}(P, Q, K) = g_{\alpha\beta} (P - Q)_\mu + g_{\beta\mu} \left( Q - K + \frac{1}{\xi} P \right)_\alpha + g_{\mu\alpha} \left( K - P - \frac{1}{\xi} Q \right)_\beta. \quad (13)$$

Calculating the transverse function  $\Pi^T(p_0 = 0, p^2 = \kappa^2)$  we find using the bare propagators in Eq.(12) the following result for  $\beta_\tau^{(0)}$  of Eq.(11):

$$\beta_\tau^{(0)} = \frac{g^3}{8\pi^2} \frac{N\pi^2}{16} (21 + 6\xi + \xi^2) \frac{\tau}{\kappa}. \quad (14)$$

For  $\xi = 0$  and  $\xi = 1$  this coincides with [14] and [28], respectively, and confirms the conjecture in [15] that the difference between their result and that of [28] is due to the gauge choice.

The general one-loop calculation with effective propagators and vertices is difficult but in our case there are some simplifications. First we consider the external energy to be zero, and in the ITF only the  $n = 0$  internal modes need resummation since all other modes are hard. The effective propagators are thus only needed for zero energy and then they take the simple form

$$D_{\mu\nu}^{*ab}(0, \mathbf{p}) = -i\delta^{ab} \left[ -\frac{1}{p^2 + m_T^2} (-\delta_{ij} + \frac{p_i p_j}{p^2}) - \frac{1}{p^2 + m_L^2} \delta_{\mu 0} \delta_{\nu 0} + \xi \frac{p_i p_j}{p^4} \right], \quad (15)$$

where  $m_T$  and  $m_L$  are transverse (magnetic) and longitudinal (electric) masses respectively. The longitudinal electric mass  $m_L^2 = \frac{1}{3}g^2 N \tau^2$  comes from the one-loop hard thermal loops, but the transverse magnetic mass  $m_T$  is zero perturbatively. Here we try to estimate its effects by hand by inclusion of the term  $m_T \sim O(g^2 \tau)$  in the propagator as a crude approximation to the true (non-perturbative) situation. Only the



pure gauge boson diagrams are effected by the resummation. The correction to e.g. the tadpole diagram is

$$\frac{T}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \left( \gamma_{\mu\nu\alpha\beta}^{*abcd} D_{\alpha\beta}^{*cd}(K) - \gamma_{\mu\nu\alpha\beta}^{abcd} D_{\alpha\beta}^{cd}(K) \right), \quad (16)$$

where the star  $*$  denotes effective vertices and propagators. Each of the two terms in Eq.(16) is quadratically divergent and receives contribution from all Matsubara frequencies at high  $T$ . The difference however is only logarithmically divergent and to get the leading  $\tau/\kappa$  term only the  $n = 0$  mode is needed. It then follows that  $\gamma_{\mu\nu\alpha\beta}^{*abcd}$  is only needed for zero external energy and then it reduces to the bare vertex. Similar simplifications can be done for the bubble diagram. The degree of divergence is reduced by two when subtracting the unresummed result and that is enough for using the  $n = 0$  approximation. We write the additional contribution to  $\Pi^T$  from non-zero  $m_L$  and  $m_T$  like

$$\delta\Pi^T(z_L, z_T) = \delta_L\Pi^T(z_L) + \delta_T^{(\xi=1)}\Pi^T(z_T) + (1 - \xi)\delta_T^{(\xi \neq 1)}\Pi^T(z_T), \quad (17)$$

where  $z_L = m_L/\kappa$  and  $z_T = m_T/\kappa$ . The explicit expressions turn out to be

$$\begin{aligned} \delta_L\Pi^T &= \frac{g^2 N}{4\pi^2} T\kappa \left\{ -\frac{\pi}{2} z_L + \frac{\pi^2}{2} z_L^2 - \frac{\pi}{4} [1 + 4z_L^2] \arctan(2z_L) \right\}, \\ \delta_T^{(\xi=1)}\Pi^T &= \frac{g^2 N}{4\pi^2} T\kappa \left\{ -\frac{\pi}{8} \frac{5 + 4z_T^2}{z_T} + \frac{3\pi^2}{4} z_T^2 \right. \\ &\quad \left. - \frac{\pi}{16} \frac{(4z_T^2 + 1)(8z_T^4 - 12z_T^2 + 1)}{z_T^4} \arctan(2z_T) \right. \\ &\quad \left. + \frac{\pi}{8} \frac{4z_T^6 + 3z_T^4 - 4z_T^2 + 1}{z_T^4} \arctan(z_T) \right\}, \\ \delta_T^{(\xi \neq 1)}\Pi^T &= \frac{g^2 N}{4\pi^2} T\kappa \left\{ -\frac{\pi}{4} \frac{2z_T^2 - 1}{z_T} - \frac{\pi^2}{4} z_T^2 - \frac{\pi}{4} \frac{1 + z_T^2 - 2z_T^4}{z_T^2} \arctan(z_T) \right\}. \end{aligned} \quad (18)$$

In this the following integrals have been used:

$$\begin{aligned} \int_0^\infty \frac{dx}{x} \ln \left( \frac{x+1}{x-1} \right)^2 &= \pi^2, \\ \int_0^\infty \frac{x dx}{x^2 + z^2} \ln \left( \frac{x+1}{x-1} \right)^2 &= \pi^2 - 2\pi \arctan(z), \\ \int_0^\infty \frac{dx}{x} \ln \left[ \frac{(x+1)^2 + z^2}{(x-1)^2 + z^2} \right] &= \pi^2 - 2\pi \arctan(z), \\ \int_0^\infty \frac{x dx}{x^2 + z^2} \ln \left[ \frac{(x+1)^2 + z^2}{(x-1)^2 + z^2} \right] &= \pi^2 - 2\pi \arctan(2z). \end{aligned} \quad (19)$$

The limit of  $\delta\Pi^T$  for small  $z$  is given by

$$\delta_L\Pi^T \simeq \frac{g^2 N}{4\pi^2} T\kappa \left[ -\frac{\pi}{4} z_L + \frac{\pi^2}{8} z_L^2 - \frac{\pi}{3} z_L^3 + \dots \right],$$

$$\begin{aligned}
\delta_T^{(\xi=1)} \Pi^T &\simeq \frac{g^2 N}{4\pi^2} T \kappa \left[ \frac{10\pi}{3} z_T + \frac{3\pi^2}{4} z_T^2 - \frac{91\pi}{15} z_T^3 + \dots \right] , \\
\delta_T^{(\xi \neq 1)} \Pi^T &\simeq \frac{g^2 N}{4\pi^2} T \kappa \left[ -\frac{2\pi}{3} z_T - \frac{\pi^2}{4} z_T^2 + \frac{8\pi}{15} z_T^3 + \dots \right] , 
\end{aligned} \tag{20}$$

while for large  $z$  we find

$$\begin{aligned}
\delta_L \Pi^T &\simeq \frac{g^2 N}{4\pi^2} T \kappa \left[ -\frac{\pi^2}{8} + \frac{\pi}{12z_L} - \frac{\pi}{240z_L^3} + \dots \right] , \\
\delta_T^{(\xi=1)} \Pi^T &\simeq \frac{g^2 N}{4\pi^2} T \kappa \left[ \frac{23\pi^2}{16} - \frac{13\pi}{6z_T} + \frac{47\pi}{120z_T^3} + \dots \right] , \\
\delta_T^{(\xi \neq 1)} \Pi^T &\simeq \frac{g^2 N}{4\pi^2} T \kappa \left[ -\pi z_T - \frac{\pi^2}{8} + \frac{2\pi}{3z_T} - \frac{\pi^2}{8z_T^2} + \dots \right] . 
\end{aligned} \tag{21}$$

It is worth noting that  $\delta_L \Pi^T$  is independent of the gauge parameter  $\xi$  and that it contains terms that are potentially dominant for large  $z_L$ . However, it turns out that the leading terms cancel between the tadpole and the bubble diagram, and that  $\delta_L \Pi^T$  only contributes to  $c_0$  (and not to  $c_1$ ) in Eq.(11). When added to the bare one-loop result  $\beta_\tau^{(0)}$  of Eq.(14) we find for  $\xi = 0$

$$\beta_\tau^{(0)} + \beta_\tau^L \simeq \frac{g^3}{8\pi^2} N \frac{\tau}{\kappa} \frac{23\pi^2}{16} . \tag{22}$$

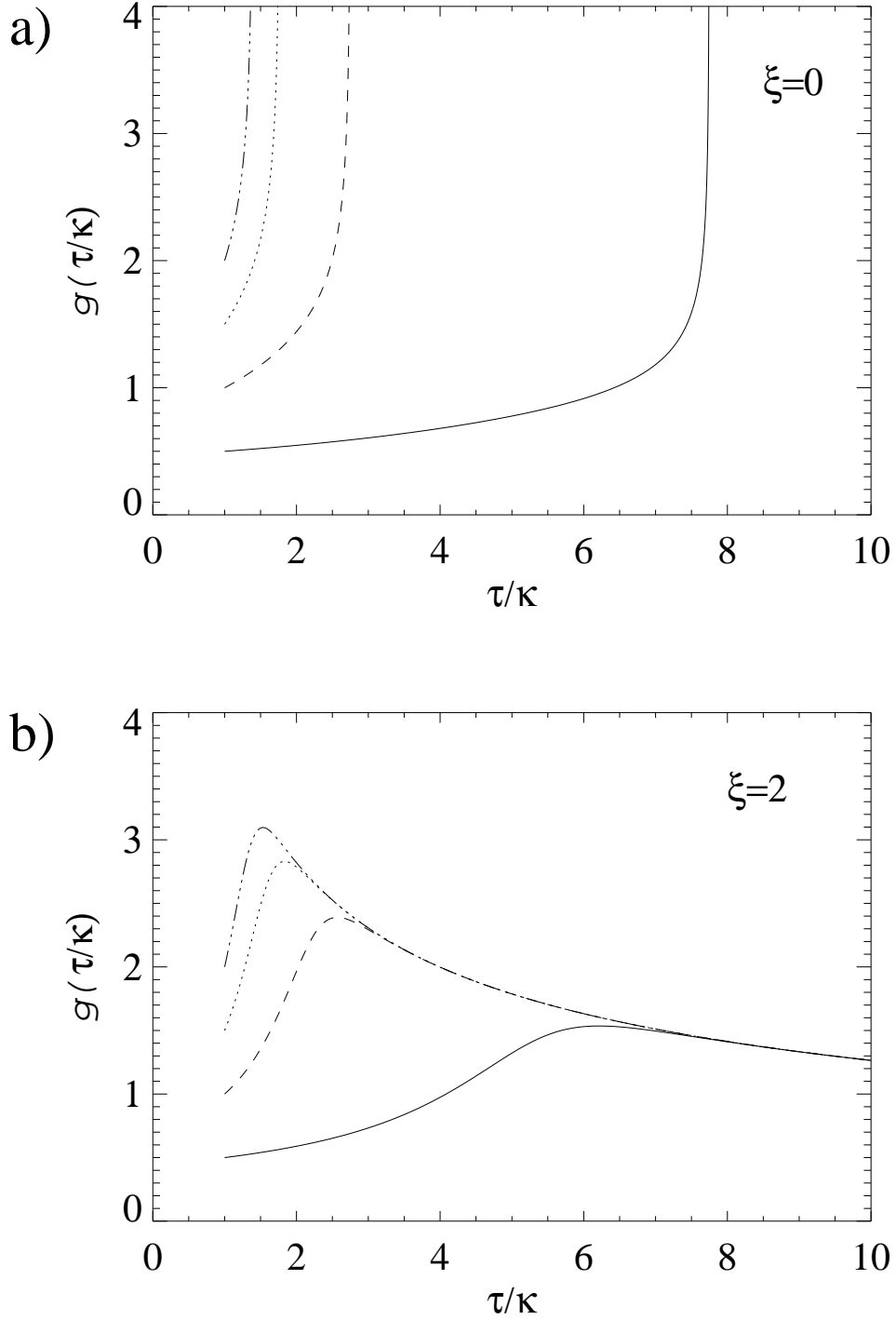
The resummation has not changed the sign but the quantitative result to this order, showing that it was necessary to include these effects for a consistent calculation. The results of [10, 14, 15] are in this sense incomplete, but the general conclusions are correct since the sign remains unchanged. They could have been drastically changed if, for instance, the linear  $m_L$  had come non-vanishing and negative (see Section 2).

When including  $m_T$  the large  $z$  limit gives

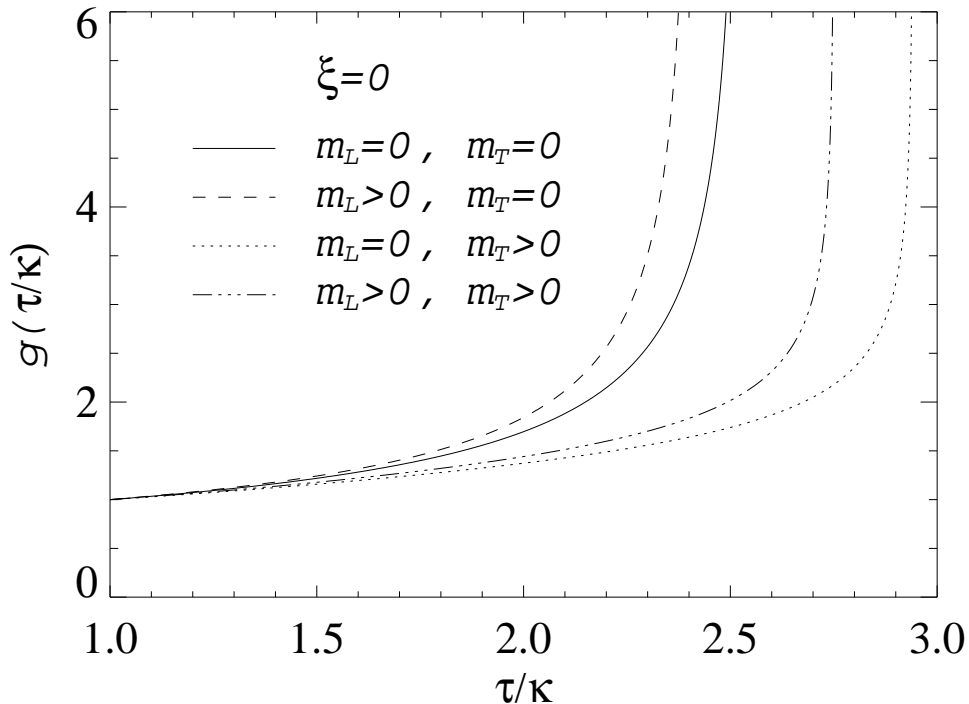
$$\beta_\tau \simeq \frac{g^3}{8\pi^2} N \frac{\tau}{\kappa} \left( \frac{(\xi+3)^2 + 12}{16} \pi^2 + \frac{\pi^2}{8} + \left[ (1-\xi) \left( \pi \frac{m_T}{\kappa} + \frac{\pi^2}{8} \right) - \frac{23}{16} \pi^2 \right] \right) , \tag{23}$$

where everything inside the [ ]-parentheses comes from the inclusion of  $m_T$ . In the Landau gauge  $\beta_\tau$  is positive even when including  $m_T$ . Even though it is possible to choose a gauge with a large enough  $\xi$  in order to stabilize the running of  $\alpha_s$  it seems rather artificial since we only can argue in favour of the  $\xi = 0$  gauge from the Vilkovisky–DeWitt approach.

We have numerically solved the RGE in the high  $\tau$  limit at a fixed momentum scale  $\kappa$  (i.e. neglecting the vacuum contribution and expanding in  $\kappa/\tau$ ) but using the exact dependence on  $m_L$  and  $m_T$ . The result is presented in Figs.(1, 2). In the  $\xi = 0$  gauge (Fig.(1a)) the coupling constant diverges at a finite temperature, just like without



**Figure 1:** The running coupling constant in a  $SU(2)$  Yang-Mills theory for the initial conditions  $g(1) = 0.5, 1.0, 1.5$  and  $2.0$ . Only the thermal contribution is included. The hard thermal loops are resummed and a magnetic mass  $m_T = cg^2\tau$  with  $c = 0.24$ , is included. The value of  $c$  is taken from the numerical simulations in [22]. In the Landau gauge,  $\xi = 0$  (Fig.(1a)), the coupling diverges at a finite temperature and perturbation theory breaks down. In the  $\xi = 2$  gauge (Fig.(1b)) the contribution from non-zero  $m_T$  prevents the coupling from diverging.



**Figure 2:** The effects of including or excluding  $m_L$  and  $m_T$  in the  $\beta_\tau$ -function. The parameters are the same as in Fig.(1) with  $\xi = 0$  and  $g(1) = 1$ .

resummation. If we choose  $\xi > 1$ , e.g.  $\xi = 2$  as in Fig.(1b), the contribution from  $m_T$  changes the sign of  $\beta_\tau$  for large  $\tau/\kappa$ . To see the effect of resummation in the  $\xi = 0$  gauge we have computed  $g(\tau/\kappa)$  with and without the contribution from  $m_L$  and  $m_T$  (see Fig.(2)). We find that the qualitative behaviour is not drastically changed by the resummation. The inclusion of  $m_L$  has a tendency to increase the growth of  $g(\tau/\kappa)$  while  $m_T$  pushes the divergence to a higher temperature.

## 4 DISCUSSION

The problems associated with a consistent calculation to this order of the  $\beta$ -function concerning the “wrong sign” and the gauge dependence are reminiscent of the early one-loop bare calculations of the gluon damping constant at rest. Such calculations also gave the “wrong sign”, in that the modes were anti-damped, and the results were also gauge parameter dependent. The use of the Vilkovisky–DeWitt effective action to address the problem of gauge dependence in this case did not resolve the problem completely, as the damping constant calculated in this formalism still had the wrong sign. Indeed, there were also arguments for choosing the background field Feynman gauge

$\xi = 1$  as the “preferred” gauge based on the gauge invariant propagator of Cornwall [29], but this too gave the wrong sign for the bare one-loop damping constant and was quantitatively different than the Vilkovisky–DeWitt gauge  $\xi = 0$ . The resolution to these problems was later supplied by the Braaten–Pisarski resummation scheme [23], where a gauge invariant and positive result is obtained to first order. Lessons from this could be drawn for this calculation of the  $\beta$ -function. The results presented here indicate that higher-loop effects can change the result quantitatively, but the particular corrections considered here were not enough to resolve the problems of the wrong sign and of gauge dependence. This may mean that if the renormalization group equations in this form are to provide a useful tool a further resummation below the soft  $O(gT)$  scale is needed to do a consistent calculation for the  $\beta$ -function; as argued in [15], the fact that the combination  $\kappa/\tau$  appears means that a large temperature expansion is in a sense the same as probing the infrared behaviour. The need for such a further resummation in this context can also be seen when the simultaneous running of the coupling constant with temperature  $\tau$  and momentum scale  $\kappa$  is investigated; from Eqs.(6,7), we see that with the particular resummation investigated here the integrability condition

$$\tau \frac{d}{d\tau} \beta_\kappa = \kappa \frac{d}{d\kappa} \beta_\tau \quad (24)$$

is not automatically satisfied. This particular problem could be cured in a somewhat *ad hoc* manner by having the  $T$ -derivative in Eq.(6) act not only on the explicit  $T$  dependence arising from the Matsubara frequency sum but also on the implicit  $T$  dependence of the masses  $m_L$  and  $m_T$ . Doing so changes the results presented here only slightly quantitatively, however. One might thus expect that an improved resummation scheme, as well as addressing the problems of the sign of the  $\beta$ -function and of gauge dependence, would also yield an integrable set of equations for  $g(\tau, \kappa)$ . The need for such a resummation beyond that of Braaten–Pisarski has also been recognized in the calculations of the damping rates of moving particles and of the production rates of soft photons [30]. Whether such a scheme can be developed and can be used to give tractable results remains to be seen.

## Acknowledgments

We are grateful for the hospitality at ENSLAPP, Annecy, where most of this work was carried out, and for stimulating discussions with the staff and visitors, especially P. Aurenche. P. E. wishes to thank The Swedish Institute for financial support. R. K. thanks NSERC of Canada and CNRS of France for financial support.

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